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1980 J. Phys. A: Math. Gen. 13 1925

(http://iopscience.iop.org/0305-4470/13/6/015)

J. Phys. A: Math. Gen. 13 (1980) 1925–1945. Printed in Great Britain

Reduction of enveloping algebras of low-rank groups†

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Received 23 August 1979, in final form 2 January 1980

Abstract. We find the generating function for group tensors contained in the enveloping algebra of each simple compact group of rank three or less. The generating function depends on dummy variables which carry, as exponents, the degrees and representation labels of the tensors; it suggests an integrity basis, a finite number of elementary tensors, in terms of which all can be expressed as stretched tensor products.

1. Introduction

Much work has been done recently toward understanding the structure of the enveloping algebras (polynomials in the generators) of simple compact groups.

It has long been known that for a group G of rank l there are just l independent invariant polynomials in the generators, or Casimir invariants (Samuelson 1941, Borel and Chevalley 1955). Recently, simplified derivations of their eigenvalues have been given by Okubo (1977) and Edwards (1978).

A theorem due to Kostant (1963) states that the number p_λ of independent λ tensors in the enveloping algebra is equal to the number of states of zero weight in the representation (λ) (here (λ) is any irreducible representation other than the scalar one). Kostant also shows that the highest degree of a λ tensor (modulo multiplying it by Casimir operators) is the sum of the coefficients of the simple roots in the highest weight of (λ); in terms of the conventional (Cartan) labels λ, this highest degree is as follows:

$$\begin{array}{llll} \mathrm{SU}(3){:} & \lambda_1+\lambda_2, & \mathrm{O}(5){:} & \tfrac{3}{2}\lambda_1+2\lambda_2, \\ \mathrm{G}_2{:} & 3\lambda_1+5\lambda_2, & \mathrm{SU}(4){:} & \tfrac{3}{2}\lambda_1+2\lambda_2+\tfrac{3}{2}\lambda_3, \\ \mathrm{Sp}(6){:} & \tfrac{5}{2}\lambda_1+4\lambda_2+\tfrac{9}{2}\lambda_3, & \mathrm{O}(7){:} & 3\lambda_1+5\lambda_2+3\lambda_3. \end{array} \tag{1.1}$$

In this paper we construct the generating function $\mathscr{G}$ for tensors in the enveloping algebra of each rank-two and rank-three group. $\mathscr{G}$ is a rational function of $l+1$ dummy variables $U, A_1, \ldots, A_l$. When it is expanded in a power series,

$$\mathscr{G}=\sum_u U^u \sum_\lambda c_{u\lambda}\Lambda^\lambda, \qquad \Lambda^\lambda=\prod_{i=1}^{l}\Lambda_i^{\lambda_i}, \tag{1.2}$$

U carries the degree u and Λ_i carries the representation label λ_i as exponents. A term $U^u c_{u\lambda}\Lambda^\lambda$ in (1.2) says that the number of λ tensors of degree u is $c_{u\lambda}$.

† Supported by the Natural Sciences and Engineering Research Council of Canada, and by the Ministère de l'Education du Québec.

The generating function does more than count the tensors. It suggests an integrity basis, a finite set of elementary tensors, in terms of which every tensor in the enveloping algebra can be expressed as a stretched product (representation labels additive).

The generating function $\mathscr{G}$ is a fraction, or sum of fractions, whose denominator factors have the form $1-X$; the X and the numerator terms Y are products of powers of U and Λ_i. The elementary tensors correspond to the X and to certain of the Y (the other Y, if any, are products of powers of the elementary tensors). Once the elementary X and Y are identified, it is straightforward, by standard methods, to determine the algebraic form of the tensors they denote. The absence of a particular product of elementary tensors implies a corresponding syzygy (relation) involving it.

The number of labels needed to specify a particular term in the enveloping algebra is r, the order (number of generators) of the group G. Subtracting $\frac{1}{2}(r-l)$, the number of internal G labels (Racah 1951), we obtain $\frac{1}{2}(r+l)$ as the number of functionally independent elementary tensors; $\frac{1}{2}(r+l)$ is thus the (maximum) number of denominator factors in each term of $\mathscr{G}$.

Apart from its primary purpose—to decompose the enveloping algebra of a group G—our generating function has other uses. Without the denominator factors which correspond to Casimir invariants, and with $U=1$, it is a generating function for the number of states of zero weight in representations of G. The generating function for the branching rules to a subgroup $\mathrm{H}\subset\mathrm{G}$ may be substituted into our generating function to obtain a generating function for subgroup tensors in the enveloping algebra of the group; with the dummies carrying the subgroup representation labels set equal to zero, one obtains a generating function for subgroup scalars—labelling operators. The group generators are no longer independent when acting on restricted representations of G—say with some of the Cartan labels zero; the relations can be described as the vanishing of group tensors in the enveloping algebra and by the corresponding collapse of the generating function for tensors.

In § 2 we describe the methods by which the generating functions may be determined. In this connection the group–subgroup characteristic function is defined. Section 3 is devoted to the generating functions for SU(3) and O(5), § 4 to SU(4) and G_2, and § 5 to Sp(6) and O(7). Section 6 describes the checks made on the generating functions. Section 7 contains some closing remarks.

2. How the generating functions are determined

In determining the generating function for tensors in the enveloping algebra of a group G, the operator properties of the generators may be ignored. Their order in a product does not affect its transformation properties under G, and in any case the commutation rules may be used to reduce any product to possibly lower-degree polynomials which are symmetric as to order. Symmetric polynomials correspond one-to-one to polynomials in c-number variables representing the generators. Hence our problem is to find the generating function for polynomial tensors in the components of a tensor which transforms by the adjoint representation of G.

Gaskell *et al* (1978) (we will refer to this paper as I) describe a general method for constructing the generating function for polynomial tensors based on any tensor of a compact group G. Although it is directly applicable to our problem, the tedium of the method increases rapidly with the number of generators. There are two devices for simplifying the work, each of more or less general applicability. The first makes use of a larger group G′ in the chain $SU(r) \supset G' \supset G$ (r is the order of G); the second involves working through a subgroup H of G. A special relationship between the enveloping algebras of G_2 and SU(4) allows the calculation of the former's generating function from the latter's; the connection is explained in § 4.

The tensors of degree u in the generators of G are precisely the multiplets contained in the representation $(u0\ldots0)$ of SU(r). Hence the generating function (1.2) is that for the branching rules $SU(r) \supset G$, restricted to one-rowed representations of SU(r). The calculation is simplified if a group G′ can be inserted in the chain $SU(r) \supset G' \supset G$. One finds the generating functions for $SU(r) \supset G'$ and for $G' \supset G$ and substitutes the latter in the former. Thus for O(5) one may use the chain $SU(10) \supset SU(5) \supset O(5)$; the embedding is such that $(10\ldots0)$ of SU(10) contains (0100) of SU(5) which contains (20) of O(5). An alternative chain is $SU(10) \supset SU(4) \supset O(5)$ with the embedding $(10\ldots0) \supset (200) \supset (20)$. Similarly for SU(4) the chain $SU(15) \supset SU(6) \supset SU(4)$ is available with the embedding $(10\ldots0) \supset (01000) \supset (101)$. For O(7) one may use $SU(21) \supset SU(7) \supset 0(7)$ with $(10\ldots0) \supset (010000) \supset (010)$. For Sp(6) there is $SU(21) \supset SU(6) \supset Sp(6)$ with $(10\ldots0) \supset (20000) \supset (200)$. The generating functions for the relevant group–subgroup branching rules are described in §§ 3, 4 and 5. The procedure for substituting one in another is detailed by Patera and Sharp (1980).

The alternative approach is to work through a subgroup H of G. The generators of G form a reducible tensor of H. It may be relatively easy to construct the generating function for polynomial H tensors in the components of that reducible tensor. Under certain circumstances it may be possible to convert that generating function into the corresponding (and desired) generating function for G tensors. A necessary tool in converting a generating function for subgroup tensors into the corresponding generating function for group tensors is the group–subgroup characteristic function, to which the remainder of this section is devoted.

The group–subgroup characteristic function is a generalisation of Weyl's (1925) characteristic function, to which it reduces when the subgroup is the Cartan subgroup, $U(1)\times\ldots\times U(1)$ (l times), whose representation labels are the components of the weight. Apart from its role in converting subgroup to group generating functions, it is useful as a way of presenting branching rules.

Let G be a simple compact group and H its semisimple or reductive subgroup. The subgroup content of an irreducible representation (λ) of G may be written

$$\chi_\lambda^{H}(N)=\sum_\nu c_{\lambda\nu}N^\nu, \qquad N^\nu=\prod_{i=1}^{l_H} N_i^{\nu_i}. \tag{2.1}$$

$(\nu)=(\nu_1,\ldots,\nu_{l_H})$ are the representation labels of H, and N are dummy variables carrying those labels as exponents; $c_{\lambda\nu}$ is the multiplicity of the subgroup representation (ν) in the group representation (λ).

In the Appendix, the group–subgroup characteristic $\xi_\lambda^{H}(N)$ is defined, and it is proved that in terms of it the subgroup content may be written

$$\chi_\lambda^{H}(N)=\xi_\lambda^{H}(N)/\xi_0^{H}(N). \tag{2.2}$$

$\xi_0^{H}(N)$ may be evaluated straightforwardly from its definition (A1) in each case of interest. In principle $\xi_\lambda^{H}(N)$ may also be evaluated from the definition; however, it may be easier to evaluate $\chi_\lambda^{H}(N)$ in the form (2.2) from the generating function for $G \supset H$ branching rules, if it is known, and read off $\xi_\lambda^{H}(N)$. We give explicit examples.

The group–subgroup generating function for $SU(3) \supset SU(2) \times U(1)$ is known to be (Sharp and Lam 1969)

$$\mathscr{F}(\Lambda_1, \Lambda_2; N_1, N_2) = [(1-\Lambda_1 N_1 N_2)(1-\Lambda_1 N_2^{-2})(1-\Lambda_2 N_1 N_2^{-1})(1-\Lambda_2 N_2^2)]^{-1}, \tag{2.3}$$

where Λ_1, Λ_2 carry the SU(3) representation labels as exponents, N_1 carries twice the isospin, and N_2 three times the hypercharge. The $SU(2) \times U(1)$ content of the SU(3) representation (λ_1, λ_2) is the coefficient of $\Lambda_1^{\lambda_1}\Lambda_2^{\lambda_2}$ in the expansion of (2.3); this can be evaluated by taking appropriate residues of (2.3):

$$\begin{aligned}\chi^{\mathrm{H}}_{\lambda_1\lambda_2}(N_1, N_2) &= \sum \mathrm{Res}_{\Lambda_1\Lambda_2} \Lambda_1^{\lambda_1-1}\Lambda_2^{\lambda_2-1}\mathscr{F}(\Lambda_1^{-1}, \Lambda_2^{-1}; N_1, N_2)\\ &= \xi^{\mathrm{H}}_{\lambda_1\lambda_2}(N_1, N_2)/\xi^{\mathrm{H}}_{00}(N_1, N_2),\end{aligned} \tag{2.4}$$

with

$$\begin{aligned}\xi^{\mathrm{H}}_{\lambda_1\lambda_2}&(N_1, N_2)\\ &= N_1^{\lambda_2+1}N_2^{-2\lambda_1-\lambda_2-3} + N_1^{\lambda_1+1}N_2^{\lambda_1+2\lambda_2+3}\\ &\quad - N_1^{\lambda_1+\lambda_2+2}N_2^{\lambda_1-\lambda_2} - N_2^{2\lambda_2-2\lambda_1},\end{aligned} \tag{2.5a}$$

$$\xi^{\mathrm{H}}_{00}(N_1, N_2) = (N_1N_2 - N_2^{-2})(N_2^2 - N_1N_2^{-1}). \tag{2.5b}$$

The Λ_1, Λ_2 residues in (2.4) are those at poles inside circles a little greater than unity in radius; $|N_1|$ is small compared with unity, and $|N_2|$ equal to unity.

Similarly, from the $SU(3) \supset O(3)$ generating function

$$\mathscr{F}(\Lambda_1, \Lambda_2; N) = (1+\Lambda_1\Lambda_2 N)[(1-\Lambda_1 N)(1-\Lambda_1^2)(1-\Lambda_2 N)(1-\Lambda_2^2)]^{-1},$$

we find the $SU(3) \supset O(3)$ characteristic function

$$\xi^{\mathrm{H}}_{\lambda_1\lambda_2}(N) = (1-N^{\lambda_1+1})(1-N^{\lambda_2+1}) + N^{-1}(1-N)^2\delta(\lambda_1, \lambda_2), \tag{2.6a}$$

$$\xi^{\mathrm{H}}_{00} = (1-N)^2(1+N^{-1}). \tag{2.6b}$$

$\delta(\lambda_1, \lambda_2)$ is unity if λ_1, λ_2 are both even, and zero otherwise. N carries the angular momentum quantum number as exponent.

From the $O(5) \supset SU(2) \times SU(2)$ generating function

$$\mathscr{F}(\Lambda_1, \Lambda_2; N_1, N_2) = [(1-\Lambda_1 N_1)(1-\Lambda_1 N_2)(1-\Lambda_2)(1-\Lambda_2 N_1 N_2)]^{-1},$$

we find the $O(5) \supset SU(2) \times SU(2)$ characteristic function

$$\xi^{\mathrm{H}}_{\lambda_1\lambda_2}(N_1, N_2) = N_2^{\lambda_1+\lambda_2+2}N_1^{\lambda_2+1} - N_2^{\lambda_2+1}N_1^{\lambda_1+\lambda_2+2} + N_1^{\lambda_1+1} - N_2^{\lambda_1+1}, \tag{2.7a}$$

$$\xi^{\mathrm{H}}_{00}(N_1, N_2) = (N_1N_2 - 1)(N_2 - N_1). \tag{2.7b}$$

From the $G_2 \supset SU(3)$ generating function (Sharp and Lam 1969, and I)

$$\begin{aligned}\mathscr{F}(\Lambda_1, \Lambda_2; &N_1, N_2)\\ &= [(1-\Lambda_1 N_1)(1-\Lambda_1 N_2)(1-\Lambda_2 N_1)(1-\Lambda_2 N_2)]^{-1}\\ &\quad \times \{(1-\Lambda_1)^{-1} + \Lambda_2 N_1 N_2 (1-\Lambda_2 N_1 N_2)^{-1}\},\end{aligned} \tag{2.8}$$

we find the $G_2 \supset SU(3)$ characteristic function

$$\begin{aligned}\xi^{H}_{\lambda_1\lambda_2}(N_1, N_2) \\ &= N_1^{\lambda_2+1}N_2^{\lambda_1+\lambda_2+2} - N_1^{\lambda_1+\lambda_2+2}N_2^{\lambda_2+1} + N_2^{\lambda_2+1} \\ &\quad - N_1^{\lambda_2+1} + N_1^{\lambda_1+\lambda_2+2} - N_2^{\lambda_1+\lambda_2+2},\end{aligned} \tag{2.9a}$$

$$\xi^{H}_{00}(N_1, N_2) = (N_2 - N_1)(N_1 - 1)(N_2 - 1). \tag{2.9b}$$

We now turn to the problem of converting a generating function for subgroup tensors into the corresponding generating function for group tensors. The group-subgroup characteristic function $\xi^{H}_{\lambda}(N)$ is a linear combination of terms $\Pi_i N_i^{p_i}$, whose exponents p_i depend linearly on the λ_j; the dummies N_i are defined so that the coefficients of the λ_j are all integers. Pick out one term of $\xi^{H}_{\lambda}(N)$ and represent it by the point in l_H-dimensional space whose Cartesian components are its exponents p_i. Associated with the term is the sector in which its point lies when the representation labels λ_j take all possible values. To be useful for our present purpose $\xi^{H}_{\lambda}(N)$ should contain a term—to be called the *distinctive* term—which satisfies two criteria. Its sector should not be overlapped by the sector of any other term, and its exponents $\bar{p}_i$ must determine the representation labels λ_j. For $SU(3) \supset O(3)$ or any pair in which the subgroup has lower rank than the group, the group–subgroup characteristic function contains no distinctive term. But for all group–subgroup pairs which we have examined with $l_H = l_G$, including $SU(3) \supset SU(2) \times U(1)$, $O(5) \supset SU(2) \times SU(2)$, $SU(4) \supset SU(3) \times U(1)$, $G_2 \supset SU(3)$, $G_2 \supset O(4)$, the group–subgroup characteristic contains at least one distinctive term; we conjecture that this is the case whenever $l_H = l_G$. The distinctive term is the first term on the right-hand side of each of equations (2.5*a*), (2.7*a*) and (2.9*a*).

Consider a generating function $\mathscr{F}(N)$ for tensors of the subgroup H of a group G. We assume that the tensors are those contained in complete tensors of G and write

$$\mathscr{F}(N) = \sum_{\lambda} \chi^{H}_{\lambda}(N) c_{\lambda}, \tag{2.10}$$

where $\chi^{H}_{\lambda}(N)$ is the subgroup content of the irreducible representation (λ) of G (see equation (2.1)); c_λ is the multiplicity of (λ) in $\mathscr{F}$, and may depend on other dummy variables such as U in (1.2). Multiply (2.10) by $\xi^{H}_{0}(N)$; because of (2.2) we obtain

$$\xi^{H}_{0}(N)\mathscr{F}(N) = \sum_{\lambda} \xi^{H}_{\lambda}(N) c_{\lambda}. \tag{2.11}$$

We now assume that the criterion of the preceding paragraph is satisfied by $G \supset H$. Multiply (2.11) by $\Pi_i N_i^{-\bar{p}_i - 1} \Pi_j \Lambda_j^{\lambda_j}$ and sum over λ_j from 0 to ∞; the sums are geometric and may be done explicitly. Finally add all residues at poles of $N_1, \ldots, N_l$ inside their unit circles. The result is the desired generating function $\mathscr{G}(\Lambda)$ for G tensors. Examples of the procedure for $SU(3) \supset SU(2) \times U(1)$ and $O(5) \supset SU(2) \times SU(2)$ are given in § 3.

3. SU(3) and O(5)

Before dealing with the two classical rank-two groups SU(3) and O(5), we dispose of the rank-one group SU(2). The generating function for tensors in its enveloping algebra is

$$\mathscr{G}(U, \Lambda) = [(1 - U^2)(1 - U\Lambda)]^{-1}. \tag{3.1}$$

The integrity basis, which is well known, consists of two elements, a second-degree scalar and a first-degree vector, indicated in the generating function by U^2 and $U\Lambda$ respectively. The scalar is the SU(2) Casimir invariant. Any SU(2) tensor in the enveloping algebra is a stretched power of the vector multiplied by a power of the scalar.

To find the generating function for tensors in the enveloping algebra of SU(3), we make use of the SU(2)×U(1) subgroup.

The generators of SU(3) decompose under SU(2)×U(1) into a vector and scalar with U(1) labels 0 and two spinors with U(1) labels ±3. The generating function for SU(2)×U(1) tensors based on the vector and scalar is, by (3.1),

$$\mathcal{F}_1(U, N_1') = [(1-U)(1-U^2)(1-UN_1'^2)]^{-1}. \tag{3.2}$$

The factor $(1-U)^{-1}$ takes account of the scalar. The exponent of N_1', to avoid fractional exponents later, is twice the isospin. The generating function based on the spinors is (N_2 carries the U(1) label)

$$\mathcal{F}_2(U, N_1'', N_2) = [(1-U^2)(1-UN_1''N_2^3)(1-UN_1''N_2^{-3})]^{-1}. \tag{3.3}$$

The isospins of the generating functions (3.2) and (3.3) must be coupled (Patera and Sharp 1980) with the help of the SU(2) Clebsch–Gordan generating function

$$C(N_1', N_1'', N_1) = [(1-N_1'N_1'')(1-N_1'N_1)(1-N_1''N_1)]^{-1} \tag{3.4}$$

to obtain the generating function for SU(2)×U(1) tensors in the SU(3) enveloping algebra:

$$\begin{aligned}
&\mathcal{F}_3(U, N_1, N_2)\\
&\quad = \sum \operatorname{Res}_{N_1'N_1''} N_1'^{-1}N_1''^{-1}\mathcal{F}_1(U, N_1'^{-1})\mathcal{F}_2(U, N_1''^{-1}, N_2)C(N_1', N_1'', N_1)\\
&\quad = [(1-U)(1-U^2)(1-U^3N_2^6)(1-U^3N_2^{-6})(1-UN_1N_2^3)(1-UN_1N_2^{-3})]^{-1}\\
&\qquad \times[(1-UN_1^2)^{-1}(1+U^2N_1N_2^3)(1+U^2N_1N_2^{-3})+(1-U)^{-1}U^2].
\end{aligned} \tag{3.5}$$

Following the prescription of the preceding section, we multiply (3.5) by

$$(N_1N_2-N_2^{-2})(N_2^2-N_1N_2^{-1})N_1^{-2}N_2^2[(1-\Lambda_1N_2^2)(1-\Lambda_2N_1^{-1}N_2)]^{-1}$$

and take residues with respect to N_1 and N_2. The result is the desired generating function for tensors in the SU(3) enveloping algebra:

$$\begin{aligned}
&\mathcal{G}(U;\Lambda_1,\Lambda_2)\\
&\quad = \frac{1}{(1-U^2)(1-U^3)(1-U\Lambda_1\Lambda_2)(1-U^2\Lambda_1\Lambda_2)}\left(\frac{1}{1-U^3\Lambda_1^3}+\frac{U^3\Lambda_2^3}{1-U^3\Lambda_2^3}\right)\\
&\quad = \frac{1+U^2\Lambda_1\Lambda_2+U^4\Lambda_1^2\Lambda_2^2}{(1-U^2)(1-U^3)(1-U\Lambda_1\Lambda_2)(1-U^3\Lambda_1^3)(1-U^3\Lambda_2^3)}.
\end{aligned} \tag{3.6}$$

U carries the degree, and Λ_1, Λ_2 carry the SU(3) representation labels of the tensors. The generating function (3.6) is given in I and a corresponding integrity basis by Sharp (1975), in each case without details of the derivation.

The integrity basis suggested by (3.6) consists of the quadratic and cubic Casimir invariants (U^2, U^3), two octets of degrees 1 and 2 ($U\Lambda_1\Lambda_2$, $U^2\Lambda_1\Lambda_2$)and a decuplet ($U^3\Lambda_1^3$) and antidecuplet ($U^3\Lambda_2^3$) each of degree 3. A syzygy, indicated symbolically by

$$(U^3\Lambda_1^3)(U^3\Lambda_2^3)+(U^2\Lambda_1\Lambda_2)^3+(U^2\Lambda_1\Lambda_2)(U\Lambda_1\Lambda_2)^2U^2+(U\Lambda_1\Lambda_2)^3U^3=0, \tag{3.7}$$

means that $(U^3\Lambda_1^3)(U^3\Lambda_2^3)$ or $(U^2\Lambda_1\Lambda_2)^3$ is redundant and should be discarded. The two options correspond to the two forms (3.6) of the generating function (other equivalent forms are also possible).

There are several checks that can be applied to the generating function (3.6), and to the analogous generating functions to be derived in this paper; the checks are discussed in § 6.

The generating function for tensors in the O(5) enveloping algebra can be evaluated in an analogous manner, using its subgroup SU(2)×SU(2). The O(5) generators decompose under SU(2)×SU(2) into a (1, 1) quartet and the SU(2)×SU(2) generators (2, 0) and (0, 2). It is straightforward, though laborious, to construct the generating function $\mathscr{F}(U; N_1, N_2)$ for SU(2)×SU(2) tensors based on this reducible tensor. Using the O(5)⊃SU(2)×SU(2) characteristic function (2.7), we find the desired O(5) generating function

$$\mathscr{G}(U; \Lambda_1, \Lambda_2) = \frac{1+U^4\Lambda_1^2\Lambda_2}{(1-U^2)(1-U^4)(1-U\Lambda_1^2)(1-U^2\Lambda_2)(1-U^2\Lambda_2^2)(1-U^3\Lambda_1^2)}. \tag{3.8}$$

U carries the degree, and Λ_1, Λ_2 carry the O(5) representation labels of the tensors. The integrity basis consists of the quadratic and quartic Casimir invariants (U^2, U^4), two decuplets of degrees 1 and 3 ($U\Lambda_1^2$, $U^3\Lambda_1^2$), a quintet ($U^2\Lambda_2$) and 14-plet ($U^2\Lambda_2^2$) each of degree 2, and a 35-plet ($U^4\Lambda_1^2\Lambda_2$) of degree 4. The (stretched) square of the 35-plet is redundant.

Alternative derivations of (3.8) make use of the chain SU(10)⊃SU(5)⊃O(5), or SU(10)⊃SU(4)⊃O(5). The SU(10)⊃SU(5) generating function, for one-rowed representations of SU(10), is $[(1-UM_2)(1-U^2M_4)]^{-1}$, where U carries the SU(10) label (the degree) and M_2, M_4 carry the second and fourth SU(5) labels. Hence, if $\mathscr{F}(M_1, M_2, M_3, M_4; \Lambda_1, \Lambda_2)$ is the generating function for SU(5)⊃O(5) branching rules, we see that

$$\mathscr{G}(U; \Lambda_1, \Lambda_2) = \mathscr{F}(0, U, 0, U^2; \Lambda_1, \Lambda_2) \tag{3.9}$$

is the desired generating function for tensors in the O(5) enveloping algebra. The generating function $\mathscr{F}$ for SU(5)⊃O(5) branching rules is given by Patera and Sharp (1980) and gives a $\mathscr{G}(U; \Lambda_1, \Lambda_2)$, in agreement with (3.8). The SU(10)⊃SU(4) generating function, for one-rowed representations of SU(10), is $[(1-UM_1^2)(1-U^2M_2^2)(1-U^3M_3^2)(1-U^4)]^{-1}$, where U carries the SU(10) label (the degree) and M_1, M_2, M_3 carry the SU(4) labels. Hence, if $\mathscr{F}'(M_1^2, M_2^2, M_3^2; \Lambda_1, \Lambda_2)$ is the part of the SU(4)⊃O(5) generating function which is even in all the SU(4) labels, we see that

$$\mathscr{G}(U; \Lambda_1, \Lambda_2) = (1-U^4)^{-1}\mathscr{F}'(U, U^2, U^3; \Lambda_1, \Lambda_2) \tag{3.10}$$

is the desired generating function (3.8) for tensors in the O(5) enveloping algebra. The generating function for SU(4)⊃O(5) branching rules is given by Patera and Sharp (1980).

4. SU(4) and G$_2$

The generating function for tensors in the enveloping algebra of SU(4) could be evaluated by using the SU(3)×U(1) subgroup and the methods of the preceding section. We found it easier to use the chain SU(15)⊃SU(6)⊃SU(4).

The SU(15) ⊃ SU(6) generating function, for one-rowed representations of SU(15), is $[(1-UM_2)(1-U^2M_4)(1-U^3)]^{-1}$, where U carries the SU(15) label, or degree, and M_2, M_4 carry the second and fourth SU(6) labels. The SU(6) ⊃ SU(4) generating function, for representations of SU(6) in which only the second and fourth labels are non-zero, is

$$\begin{aligned}
&\mathscr{F}(M_2, M_4; \Lambda_1, \Lambda_2, \Lambda_3)\\
&\quad=[(1-M_2^2)(1-M_4^2)(1-M_2\Lambda_1\Lambda_3)(1-M_4\Lambda_1\Lambda_3)(1-M_2^2\Lambda_2^2)(1-M_4^2\Lambda_2^2)]^{-1}\\
&\quad\times\bigg(\frac{1+M_2^2M_4\Lambda_1^2\Lambda_2+M_2M_4^2\Lambda_1^2\Lambda_2+M_2^2M_4^2\Lambda_1\Lambda_2^2\Lambda_3}{(1-M_2M_4\Lambda_1\Lambda_3)(1-M_2^2M_4^2\Lambda_1^4)}\\
&\quad+\frac{M_2^2M_4\Lambda_2\Lambda_3^2+M_2M_4^2\Lambda_2\Lambda_3^2+M_2^2M_4^2\Lambda_3^4(1+M_2^2M_4^2\Lambda_1\Lambda_2^2\Lambda_3)}{(1-M_2M_4\Lambda_1\Lambda_3)(1-M_2^2M_4^2\Lambda_3^4)}\\
&\quad+\frac{M_2M_4\Lambda_1^2\Lambda_2(1+M_2^2M_4\Lambda_1^2\Lambda_2+M_2M_4^2\Lambda_1^2\Lambda_2)+M_2^3M_4^3\Lambda_1^4\Lambda_2^2}{(1-M_2M_4\Lambda_1^2\Lambda_2)(1-M_2^2M_4^2\Lambda_1^4)}\\
&\quad+\frac{M_2M_4\Lambda_2\Lambda_3^2(1+M_2^2M_4\Lambda_2\Lambda_3^2+M_2M_4^2\Lambda_2\Lambda_3^2)+M_2^3M_4^3\Lambda_2^2\Lambda_3^4}{(1-M_2M_4\Lambda_2\Lambda_3^2)(1-M_2^2M_4^2\Lambda_3^4)}\bigg).
\end{aligned} \tag{4.1}$$

M_2, M_4 carry the second and fourth SU(6) labels, and Λ_1, Λ_2, Λ_3 the SU(4) labels. Hence the desired generating function for tensors in the SU(4) algebra is

$$\begin{aligned}
&\mathscr{G}(U; \Lambda_1, \Lambda_2, \Lambda_3)\\
&\quad=(1-U^3)^{-1}\mathscr{F}(U, U^2; \Lambda_1, \Lambda_2, \Lambda_3) \qquad (4.2)\\
&\quad=[(1-U^2)(1-U^3)(1-U^4)(1-U\Lambda_1\Lambda_3)\\
&\qquad\times(1-U^2\Lambda_1\Lambda_3)(1-U^2\Lambda_2^2)(1-U^4\Lambda_2^2)]^{-1}\\
&\qquad\times\bigg(\frac{1+U^4\Lambda_1^2\Lambda_2+U^5\Lambda_1^2\Lambda_2+U^6\Lambda_1\Lambda_2^2\Lambda_3}{(1-U^3\Lambda_1\Lambda_3)(1-U^6\Lambda_1^4)}\\
&\qquad+\frac{U^4\Lambda_2\Lambda_3^2+U^5\Lambda_2\Lambda_3^2+U^6\Lambda_3^4(1+U^6\Lambda_1\Lambda_2^2\Lambda_3)}{(1-U^3\Lambda_1\Lambda_3)(1-U^6\Lambda_3^4)}\\
&\qquad+\frac{U^3\Lambda_1^2\Lambda_2(1+U^4\Lambda_1^2\Lambda_2+U^5\Lambda_1^2\Lambda_2)+U^9\Lambda_1^4\Lambda_2^2}{(1-U^3\Lambda_1^2\Lambda_2)(1-U^6\Lambda_1^4)}\\
&\qquad+\frac{U^3\Lambda_2\Lambda_3^2(1+U^4\Lambda_2\Lambda_3^2+U^5\Lambda_2\Lambda_3^2)+U^9\Lambda_2^2\Lambda_3^4}{(1-U^3\Lambda_2\Lambda_3^2)(1-U^6\Lambda_3^4)}\bigg).
\end{aligned}$$

U carries the degree, and Λ_1, Λ_2, Λ_3 the SU(4) labels of the tensors.

Inspection of (4.2) suggests an integrity basis with 17 elements (the notation is ($pabc$), where p is the degree and a, b, c the SU(4) labels): (2000), (3000), (4000), (1101), (2101), (2020), (4020), (3101), (3210), (3012), (4210), (4012), (5210), (5012), (6121), (6400), (6004). Because of syzygies, the following products of elementary tensors should be eliminated: (3101) with (3210), (3012); (3210) with (3012), (4012), (5012), (6121), (6004); (3012) with (4210), (5210), (6121), (6400); (4210) with (4012), (5012), (6121), (6004); (4012) with (5210), (6121), (6400); (5210) with (5012), (6121), (6004); (5012) with (6121), (6400); (6121) with (6400), (6004); (6400) with (6004); the squares of (4210), (4012), (5210), (5012), (6121); the products (3101)(4210)(5210) and (3101)(4012)(5012).

The generating function for tensors in the enveloping algebra of G_2 could be evaluated with the use of the SU(3) subgroup. However, it proves simpler to obtain it by exploiting an interesting relationship between the groups SU(4) and G_2 based on the fact that the subgroup SU(3) is embedded similarly in the two groups.

With the help of the SU(4) $\supset$ SU(3) generating function

$$\mathscr{F}(\Lambda_1, \Lambda_2, \Lambda_3; N_1, N_2) = [(1-\Lambda_1)(1-\Lambda_1 N_1)(1-\Lambda_2 N_1)(1-\Lambda_2 N_2)(1-\Lambda_3 N_2)(1-\Lambda_3)]^{-1}, \tag{4.3}$$

(Λ_1, Λ_2, Λ_3 carry the SU(4) labels, and N_1, N_2 the SU(3) labels) a generating function $\mathscr{H}(\Lambda_1, \Lambda_2, \Lambda_3)$ for SU(4) tensors may be converted into a generating function

$$\begin{aligned}\mathscr{J}(N_1, N_2) = {} & [(1-N_1)(1-N_2)(N_1-N_2)]^{-1}(N_1^2 N_2 \mathscr{H}(N_1, N_1, N_2) \\ & - N_1 N_2^2 \mathscr{H}(N_1, N_2, N_2) - N_1^2 \mathscr{H}(N_1, N_1, 1) + N_2^2 \mathscr{H}(1, N_2, N_2) \\ & + N_1 \mathscr{H}(1, N_1, 1) - N_2 \mathscr{H}(1, N_2, 1) + N_1 N_2 \mathscr{H}(N_1, N_2, 1) \\ & - N_1 N_2 \mathscr{H}(1, N_1, N_2))\end{aligned} \tag{4.4}$$

for SU(3) tensors.

Similarly the $G_2 \supset$ SU(3) generating function (2.8) converts a generating function $\mathscr{K}(\Lambda_1, \Lambda_2)$ for G_2 tensors into the generating function

$$\begin{aligned}\mathscr{L}(N_1, N_2) = {} & [(1-N_1)(1-N_2)(N_1-N_2)]^{-1}(N_2^2 \mathscr{K}(N_2, N_2) \\ & - N_1^2 \mathscr{K}(N_1, N_1) + N_1^2 N_2 \mathscr{K}(N_1, N_1 N_2) - N_1 N_2^2 \mathscr{K}(N_2, N_1 N_2) \\ & + N_1 \mathscr{K}(1, N_1) - N_2 \mathscr{K}(1, N_2))\end{aligned} \tag{4.5}$$

for SU(3) tensors.

Now suppose that the SU(4) generating function $\mathscr{H}(\Lambda_1, \Lambda_2, \Lambda_3)$ and the G_2 generating function $\mathscr{K}(\Lambda_1, \Lambda_2)$ are related by the fact that they generate the same SU(3) tensors. It follows from (4.4) and (4.5) that they are related by the functional equation

$$\begin{aligned} & N_2^2 \mathscr{K}(N_2, N_2) - N_1^2 \mathscr{K}(N_1, N_1) + N_1^2 N_2 \mathscr{K}(N_1, N_1 N_2) \\ & \quad - N_1 N_2^2 \mathscr{K}(N_2, N_1 N_2) + N_1 \mathscr{K}(1, N_1) - N_2 \mathscr{K}(1, N_2) \\ & = N_1^2 N_2 \mathscr{H}(N_1, N_1, N_2) - N_1 N_2^2 \mathscr{H}(N_1, N_2, N_2) \\ & \quad - N_1^2 \mathscr{H}(N_1, N_1, 1) + N_2^2 \mathscr{H}(1, N_2, N_2) + N_1 \mathscr{H}(1, N_1, 1) \\ & \quad - N_2 \mathscr{H}(1, N_2, 1) + N_1 N_2 \mathscr{H}(N_1, N_2, 1) - N_1 N_2 \mathscr{H}(1, N_1, N_2).\end{aligned} \tag{4.6}$$

Under the assumption that $\mathscr{H}(\Lambda_1, \Lambda_2, \Lambda_3)$ is symmetric in its first and last arguments, $\mathscr{H}(\Lambda_1, \Lambda_2, \Lambda_3) = \mathscr{H}(\Lambda_3, \Lambda_2, \Lambda_1)$, it can be verified that a formal solution of (4.6) for $\mathscr{K}(\Lambda_1, \Lambda_2)$ is

$$\mathscr{K}(\Lambda_1, \Lambda_2) = \mathscr{H}(\Lambda_1, \Lambda_1, \Lambda_2/\Lambda_1) + \Lambda_1^{-1} \mathscr{H}(\Lambda_1, \Lambda_2/\Lambda_1, 1). \tag{4.7}$$

The solution (4.7) suffers from the defect that its expansion contains, in general, negative powers of Λ_1. These can be eliminated by adding to $\mathscr{K}(\Lambda_1, \Lambda_2)$ an appropriate solution of the homogeneous version of (4.6). It can be verified that, for any λ_1, λ_2,

$$\mathcal{K}'(\Lambda_1, \Lambda_2) = \Lambda_1^{-\lambda_1}\Lambda_2^{\lambda_2} + \Lambda_1^{\lambda_1-2}\Lambda_2^{\lambda_2-\lambda_1+1}$$

satisfies the homogeneous equation. Thus we have the following prescription for the solution of (4.6) for $\mathcal{K}(\Lambda_1, \Lambda_2)$ which contains no negative powers: expand the right-hand side of (4.7) in powers of Λ_1 and replace each negative power $\Lambda_1^{-\lambda_1}(\lambda_1 \geq 2)$ by $-\Lambda_1^{\lambda_1-2}\Lambda_2^{-\lambda_1+1}$; drop terms in Λ_1^{-1}. Because of the form (4.7) this cannot introduce negative powers of Λ_2. The prescription can be formulated in terms of residues:

$$\mathcal{K}(\Lambda_1, \Lambda_2) = \sum \mathrm{Res}_{\Lambda_1'}\left[\frac{1}{\Lambda_1' - \Lambda_1} - \frac{\Lambda_1'}{\Lambda_2 - \Lambda_1\Lambda_1'}\right]\left[\mathcal{H}\left(\Lambda_1', \Lambda_1', \frac{\Lambda_2}{\Lambda_1'}\right) + \frac{1}{\Lambda_1'}\mathcal{H}\left(\Lambda_1', \frac{\Lambda_2}{\Lambda_1'}, 1\right)\right]. \tag{4.8}$$

The first term, $(\Lambda_1' - \Lambda_1)^{-1}$, in the first square bracket picks out the positive power part in Λ_1; the second, $\Lambda_1'(\Lambda_2 - \Lambda_1\Lambda_1')^{-1}$, replaces negative powers $\Lambda_1^{-\lambda_1}$ by $-\Lambda_1^{\lambda_1-2}\Lambda_2^{-\lambda_1+1}$ and cancels Λ_1^{-1}.

Now the generators of G_2 decompose under SU(3) into an octet, a triplet and an antitriplet, the same as SU(4), except that the SU(4) generators contain an additional scalar. It follows that, if $(1-U)\mathcal{G}(U; \Lambda_1, \Lambda_2, \Lambda_3)$, where $\mathcal{G}(U; \Lambda_1, \Lambda_2, \Lambda_3)$ is given by (4.2), is substituted for $\mathcal{H}(\Lambda_1, \Lambda_2, \Lambda_3)$ in (4.8), the result will be the desired generating functions for tensors in the G_2 enveloping algebra:

$$\begin{aligned}\mathcal{G}(U; \Lambda_1, \Lambda_2) &\\ = [(1-U^2)&(1-U^6)(1-U\Lambda_2)(1-U^2\Lambda_1^2)(1-U^3\Lambda_1)(1-U^4\Lambda_2^2)]^{-1}\\ &\times\left(\frac{1+U^5\Lambda_1^3\Lambda_2+U^6\Lambda_1^3+U^8\Lambda_1^3\Lambda_2}{(1-U^3\Lambda_1^3)(1-U^4\Lambda_1^2)}\right.\\ &+\frac{U^5\Lambda_2+U^6\Lambda_1^2\Lambda_2+U^7\Lambda_1\Lambda_2+U^{13}\Lambda_1^3\Lambda_2^2}{(1-U^4\Lambda_1^2)(1-U^5\Lambda_2)}\\ &\left.\times\frac{U^5\Lambda_1\Lambda_2+U^9\Lambda_1\Lambda_2^2+U^{12}\Lambda_2^3+U^8\Lambda_2^2}{(1-U^5\Lambda_2)(1-U^8\Lambda_2^2)}\right).\end{aligned} \tag{4.9}$$

U carries the degree, and Λ_1, Λ_2 carry the G_2 representation labels of the tensors.

The integrity basis implied by (4.9) consists of 17 elements (the notation is (pab), where p is the degree and a, b the G_2 labels): (200), (600), (101), (220), (310), (330), (402), (420), (501), (511), (531), (630), (621), (711), (802), (912), (12, 03). The following products of elementary tensors should not be used: the square or product of any two of (531), (630), (621), (511), (711), (912), (12, 03); the product of (330) with (802), (621), (711), (511), (912), (12, 03); of (420) with (802), (511), (912), (12, 03); of (501) with (531), (630); of (802) with (531), (630), (621), (711); and the product $(330)^2(501)^2$.

5. Sp(6) and O(7)

The generating function for tensors in the enveloping algebra of Sp(6) is most easily determined with the help of the chain SU(21) ⊃ SU(6) ⊃ Sp(6) for one-rowed representations of SU(21). As mentioned in § 2, the insertions are defined by $(10\ldots0)\supset(20000)\supset(200)$.

For one-rowed representations of SU(21) it is not hard to show that the $SU(21)\supset SU(6)$ branching rules are given by the generating function

$$\mathscr{F}(U; M_1, M_2, M_3, M_4, M_5)$$
$$=[(1-U^6)(1-UM_1^2)(1-U^2M_2^2)(1-U^3M_3^2)(1-U^4M_4^2)(1-U^5M_5^2)]^{-1}. \tag{5.1}$$

U carries the SU(21) label (the degree), and M_1, M_2, M_3, M_4, M_5 carry the SU(6) labels.

The generating function for $SU(6)\supset Sp(6)$ branching rules is of some interest in its own right. By examining low-lying representations of SU(6), we are led to the function

$$\begin{aligned}
&\mathscr{H}(M_1, M_2, M_3, M_4, M_5; \Lambda_1, \Lambda_2, \Lambda_3)\\
&\quad=[(1-M_1\Lambda_1)(1-M_2)(1-M_2\Lambda_2)(1-M_3\Lambda_3)(1-M_4)(1-M_4\Lambda_2)\\
&\qquad\times(1-M_5\Lambda_1)]^{-1}\{[(1-M_3\Lambda_1)(1-M_1M_3\Lambda_2)(1-M_3M_5\Lambda_2)\\
&\qquad\times(1-M_1M_3M_5\Lambda_3)]^{-1}+M_1M_4\Lambda_3[(1-M_3\Lambda_1)(1-M_1M_3\Lambda_2)\\
&\qquad\times(1-M_1M_4\Lambda_3)(1-M_1M_3M_5\Lambda_3)]^{-1}\\
&\qquad+M_2M_5\Lambda_3[(1-M_3\Lambda_1)(1-M_3M_5\Lambda_2)(1-M_2M_5\Lambda_3)(1-M_1M_3M_5\Lambda_3)]^{-1}\\
&\qquad+(M_1M_4\Lambda_3)(M_2M_5\Lambda_3)[(1-M_3\Lambda_1)(1-M_1M_4\Lambda_3)\\
&\qquad\times(1-M_2M_5\Lambda_3)(1-M_1M_3M_5\Lambda_3)]^{-1}\\
&\qquad+M_1M_5\Lambda_2[(1-M_1M_3\Lambda_2)(1-M_3M_5\Lambda_2)(1-M_1M_5\Lambda_2)\\
&\qquad\times(1-M_1M_3M_5\Lambda_3)]^{-1}+(M_1M_4\Lambda_3)(M_1M_5\Lambda_2)\\
&\qquad\times[(1-M_1M_3\Lambda_2)(1-M_1M_4\Lambda_3)(1-M_1M_5\Lambda_2)(1-M_1M_3M_5\Lambda_3)]^{-1}\\
&\qquad+(M_2M_5\Lambda_3)(M_1M_5\Lambda_2)[(1-M_3M_5\Lambda_2)(1-M_2M_5\Lambda_3)\\
&\qquad\times(1-M_1M_5\Lambda_2)(1-M_1M_3M_5\Lambda_3)]^{-1}+(M_1M_4\Lambda_3)(M_2M_5\Lambda_3)\\
&\qquad\times(M_1M_5\Lambda_2)[(1-M_1M_4\Lambda_3)(1-M_2M_5\Lambda_3)(1-M_1M_5\Lambda_2)\\
&\qquad\times(1-M_1M_3M_5\Lambda_3)]^{-1}+M_2M_4\Lambda_1\Lambda_3[(1-M_3\Lambda_1)\\
&\qquad\times(1-M_1M_4\Lambda_3)(1-M_2M_5\Lambda_3)(1-M_2M_4\Lambda_1\Lambda_3)]^{-1}\}.
\end{aligned} \tag{5.2}$$

M_1, M_2, M_3, M_4, M_5 carry the SU(6) and $\Lambda_1, \Lambda_2, \Lambda_3$ the Sp(6) representation labels. The interpretation of (5.2) in terms of an integrity basis is straightforward, and omitted here; it can be found in Couture (1980). The integrity basis, which contains 15 elements, defines SU(6) polynomial bases reduced according to the Sp(6) subgroup. While (5.2) has not been derived analytically, we are reasonably sure it is correct. For example the generating function for tensors in the Sp(6) enveloping algebra, derived from it, has been subjected to the checks described in § 6.

The generating function $\mathscr{H}$ of (5.2) must be substituted into $\mathscr{F}$ of (5.1) to obtain the desired generating function $\mathscr{G}(U; \Lambda_1, \Lambda_2, \Lambda_3)$ for tensors in the Sp(6) enveloping algebra. The form of (5.1) indicates that only the part of (5.2) which is even in all SU(6) labels is required. Let $\mathscr{H}'(M_1^2, M_2^2, M_3^2, M_4^2, M_5^2; \Lambda_1, \Lambda_2, \Lambda_3)$ be this even part; it is obtained straightforwardly from (5.2). Then the desired generating function is

$$
\begin{aligned}
&\mathscr{G}(U;\Lambda_1,\Lambda_2,\Lambda_3)\\
&\quad=(1-U^6)^{-1}\mathscr{H}'(U,U^2,U^3,U^4,U^5;\Lambda_1,\Lambda_2,\Lambda_3)\\
&\quad=[(1-U^2)(1-U^4)(1-U^6)(1-U\Lambda_1^2)(1-U^2\Lambda_2^2)(1-U^3\Lambda_3^2)(1-U^4\Lambda_2^2)\\
&\qquad\times(1-U^5\Lambda_1^2)]^{-1}\{[(1-U^3\Lambda_1^2)(1-U^2\Lambda_2)(1-U^4\Lambda_2)(1-U^9\Lambda_3^2)]^{-1}\\
&\qquad\times[1+U^3\Lambda_1\Lambda_3+U^4(\Lambda_1^2\Lambda_2+\Lambda_1\Lambda_2\Lambda_3)+U^8(\Lambda_1^2\Lambda_2+\Lambda_1\Lambda_2\Lambda_3)\\
&\qquad+U^9(\Lambda_1^2\Lambda_2^2+\Lambda_1^3\Lambda_3+\Lambda_1^2\Lambda_3^2+2\Lambda_1\Lambda_2\Lambda_3)\\
&\qquad+U^{12}(\Lambda_1^3\Lambda_2^2\Lambda_3+2\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3+\Lambda_2^2\Lambda_3^2)]\\
&\qquad+[(1-U^3\Lambda_1^2)(1-U^2\Lambda_2)(1-U^5\Lambda_3^2)(1-U^9\Lambda_3^2)]^{-1}[U^5(\Lambda_3^2+\Lambda_1\Lambda_3+\Lambda_1\Lambda_2\Lambda_3\\
&\qquad+U^8(\Lambda_1\Lambda_3^3+\Lambda_1^2\Lambda_3^2+\Lambda_1\Lambda_2\Lambda_3+\Lambda_2\Lambda_3^2+\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3+\Lambda_2^2\Lambda_3^2)\\
&\qquad+U^9(\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_2\Lambda_3^2)+U^{12}\Lambda_1\Lambda_2\Lambda_3^3\\
&\qquad+U^{13}(\Lambda_1^2\Lambda_2^2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_1^2\Lambda_3^2+\Lambda_1\Lambda_3^3+\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2\Lambda_3^3)\\
&\qquad+U^{14}(\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_1^3\Lambda_3^3+\Lambda_1^2\Lambda_3^4+\Lambda_1^2\Lambda_2^2\Lambda_3^2+\Lambda_1^2\Lambda_2\Lambda_3^2)\\
&\qquad+U^{17}(\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_1^3\Lambda_2\Lambda_3^3+\Lambda_1^3\Lambda_2^2\Lambda_3^3)+U^{18}(\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_1^3\Lambda_2\Lambda_3^3+\Lambda_1^2\Lambda_2\Lambda_3^4)\\
&\qquad+U^{21}\Lambda_1^2\Lambda_2^2\Lambda_3^4]+[(1-U^3\Lambda_1^2)(1-U^4\Lambda_2)(1-U^7\Lambda_3^2)(1-U^9\Lambda_3^2)]^{-1}\\
&\qquad\times[U^7(\Lambda_3^2+\Lambda_1\Lambda_3+\Lambda_1\Lambda_2\Lambda_3)+U^9\Lambda_2\Lambda_3^2\\
&\qquad+U^{10}(\Lambda_1\Lambda_3^3+\Lambda_1^2\Lambda_3^2+\Lambda_1\Lambda_2\Lambda_3+\Lambda_2\Lambda_3^2+\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3+\Lambda_2^2\Lambda_3^2)\\
&\qquad+U^{11}(\Lambda_1^2\Lambda_3^2+\Lambda_1\Lambda_3^3+\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2\Lambda_3^3)\\
&\qquad+U^{12}\Lambda_1\Lambda_2\Lambda_3^3+U^{15}(\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2\Lambda_3^3)\\
&\qquad+U^{16}(\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_1^3\Lambda_3^3+\Lambda_1^2\Lambda_3^4+\Lambda_1^2\Lambda_2^2\Lambda_3^2+\Lambda_1^2\Lambda_2\Lambda_3^2)\\
&\qquad+U^{17}(\Lambda_1^2\Lambda_2^2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3^3)+U^{18}(\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_1^3\Lambda_2\Lambda_3^3+\Lambda_1^2\Lambda_2\Lambda_3^4)\\
&\qquad+U^{19}(\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_1^3\Lambda_2\Lambda_3^3+\Lambda_1^3\Lambda_2^2\Lambda_3^3)+U^{21}\Lambda_1^2\Lambda_2^2\Lambda_3^4]\\
&\qquad+[(1-U^3\Lambda_1^2)(1-U^5\Lambda_3^2)(1-U^7\Lambda_3^2)(1-U^9\Lambda_3^2)]^{-1}\\
&\qquad\times[U^{12}(\Lambda_1^2\Lambda_3^2+2\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1^2\Lambda_2^2\Lambda_3^2+2\Lambda_1\Lambda_3^3+2\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_3^4)\\
&\qquad+U^{14}(\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_2\Lambda_3^4)\\
&\qquad+U^{15}(\Lambda_1^3\Lambda_3^3+2\Lambda_1^3\Lambda_2\Lambda_3^3+\Lambda_1^3\Lambda_2^2\Lambda_3^3+2\Lambda_1^2\Lambda_3^4+2\Lambda_1^2\Lambda_2\Lambda_3^4\\
&\qquad+\Lambda_1\Lambda_3^5+\Lambda_3^4+\Lambda_1\Lambda_3^3+2\Lambda_2\Lambda_3^4+2\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_2^2\Lambda_3^4+\Lambda_1\Lambda_2^2\Lambda_3^3)\\
&\qquad+U^{16}(\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_2\Lambda_3^4+\Lambda_1\Lambda_3^5+\Lambda_1^2\Lambda_3^4+\Lambda_1\Lambda_2\Lambda_3^5+\Lambda_1^2\Lambda_2\Lambda_3^4)\\
&\qquad+U^{17}(\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_1^2\Lambda_2^2\Lambda_3^4+\Lambda_1\Lambda_2\Lambda_3^5)+U^{18}\Lambda_2^2\Lambda_3^4\\
&\qquad+U^{19}(\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_1^2\Lambda_2^2\Lambda_3^4+\Lambda_1\Lambda_2\Lambda_3^5)\\
&\qquad+U^{20}(\Lambda_1\Lambda_3^5+\Lambda_1^2\Lambda_3^4+2\Lambda_1\Lambda_2\Lambda_3^5+2\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_1\Lambda_2^2\Lambda_3^5+\Lambda_1^2\Lambda_2^2\Lambda_3^4)\\
&\qquad+U^{21}(\Lambda_1\Lambda_2^2\Lambda_3^5+\Lambda_1^2\Lambda_3^6+\Lambda_1^3\Lambda_3^5)\\
&\qquad+U^{22}(\Lambda_1\Lambda_2\Lambda_3^5+\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_1\Lambda_2^2\Lambda_3^5+\Lambda_1^2\Lambda_2^2\Lambda_3^4)
\end{aligned}
$$

$$
\begin{aligned}
&+U^{23}(\Lambda_1^2\Lambda_2\Lambda_3^6+\Lambda_1^3\Lambda_2\Lambda_3^5)\\
&+U^{25}(\Lambda_1^2\Lambda_2\Lambda_3^6+\Lambda_1^3\Lambda_2\Lambda_3^5)+U^{27}(\Lambda_1^2\Lambda_2^2\Lambda_3^6+\Lambda_1^3\Lambda_2^2\Lambda_3^5)]\\
&+[(1-U^2\Lambda_2)(1-U^4\Lambda_2)(1-U^6\Lambda_2^2)(1-U^9\Lambda_3^2)]^{-1}[U^6(\Lambda_1^2\Lambda_2+\Lambda_2^2)\\
&+U^9(2\Lambda_1\Lambda_2^2\Lambda_3+\Lambda_2\Lambda_3^2+\Lambda_2^3)+U^{10}(\Lambda_1\Lambda_2^3\Lambda_3+\Lambda_1\Lambda_2^2\Lambda_3)\\
&+U^{14}(\Lambda_1\Lambda_2^3\Lambda_3+\Lambda_1\Lambda_2^2\Lambda_3)+U^{15}(\Lambda_1^2\Lambda_2^2\Lambda_3^2+2\Lambda_1\Lambda_2^3\Lambda_3+\Lambda_1^2\Lambda_2^4)\\
&+U^{18}(\Lambda_1^2\Lambda_2^3\Lambda_3^2+\Lambda_2^4\Lambda_3^2)]+[(1-U^2\Lambda_2)(1-U^5\Lambda_3^2)(1-U^6\Lambda_2^2)(1-U^9\Lambda_3^2)]^{-1}\\
&\times[U^{10}(\Lambda_1\Lambda_2\Lambda_3+\Lambda_1\Lambda_2^2\Lambda_3)+U^{11}(\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3+\Lambda_1\Lambda_2^3\Lambda_3+\Lambda_2^2\Lambda_3^2)\\
&+U^{14}(2\Lambda_2^3\Lambda_3^2+\Lambda_2^2\Lambda_3^2+\Lambda_2^4\Lambda_3^2+\Lambda_1^2\Lambda_2^2\Lambda_3^2\\
&+\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_1^2\Lambda_2^3\Lambda_3^2+2\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_2\Lambda_3^4)\\
&+U^{15}(\Lambda_1\Lambda_2^3\Lambda_3^3+\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_1^2\Lambda_2^2\Lambda_3^2+\Lambda_2^3\Lambda_3^2)+U^{18}(\Lambda_1\Lambda_2^3\Lambda_3^3+\Lambda_2^2\Lambda_3^4)\\
&+U^{19}(\Lambda_1\Lambda_2^2\Lambda_3^3+2\Lambda_1\Lambda_2^3\Lambda_3^3+\Lambda_1\Lambda_2^4\Lambda_3^3)\\
&+U^{20}(\Lambda_1^2\Lambda_2^2\Lambda_3^4+\Lambda_1^2\Lambda_2^3\Lambda_3^2+\Lambda_1^2\Lambda_2^4\Lambda_3^2+\Lambda_1\Lambda_2^3\Lambda_3^3)+U^{24}(\Lambda_1\Lambda_2^4\Lambda_3^3+\Lambda_1^2\Lambda_2^3\Lambda_3^4)]\\
&+[(1-U^4\Lambda_2)(1-U^6\Lambda_2^2)(1-U^7\Lambda_3^2)(1-U^9\Lambda_3^2)]^{-1}\\
&\times[U^8(\Lambda_1\Lambda_2\Lambda_3+\Lambda_1\Lambda_2^2\Lambda_3)+U^{13}(\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3+\Lambda_1\Lambda_2^3\Lambda_3+\Lambda_2^2\Lambda_3^2)\\
&+U^{15}(\Lambda_1^2\Lambda_2^2\Lambda_3^2+\Lambda_2^3\Lambda_3^2)+U^{16}(2\Lambda_2^3\Lambda_3^2+\Lambda_2^2\Lambda_3^2+\Lambda_2^4\Lambda_3^2+\Lambda_1^2\Lambda_2^2\Lambda_3^2\\
&+\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_1^2\Lambda_2^3\Lambda_3^2+2\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_2\Lambda_3^4)\\
&+U^{17}(\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_1\Lambda_2^3\Lambda_3^3)+U^{18}(\Lambda_1\Lambda_2^3\Lambda_3^3+\Lambda_2^2\Lambda_3^4)\\
&+U^{21}(\Lambda_1\Lambda_2^3\Lambda_3^3+\Lambda_1\Lambda_2^2\Lambda_3^3)+U^{22}(\Lambda_1^2\Lambda_2^2\Lambda_3^4+\Lambda_1^2\Lambda_2^3\Lambda_3^2+\Lambda_1^2\Lambda_2^4\Lambda_3^2+\Lambda_1\Lambda_2^3\Lambda_3^3)\\
&+U^{23}(\Lambda_1\Lambda_2^4\Lambda_3^3+\Lambda_1\Lambda_2^3\Lambda_3^3)+U^{24}(\Lambda_1^2\Lambda_2^3\Lambda_3^4+\Lambda_1\Lambda_2^4\Lambda_3^3)]\\
&+[(1-U^5\Lambda_3^2)(1-U^6\Lambda_2^2)(1-U^7\Lambda_3^2)(1-U^9\Lambda_3^2)]^{-1}\\
&\times[U^{12}(\Lambda_2\Lambda_3^2+2\Lambda_2^2\Lambda_3^2+\Lambda_2^3\Lambda_3^2)+U^{13}(\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_1\Lambda_2^2\Lambda_3^3)\\
&+U^{17}(\Lambda_1\Lambda_2\Lambda_3^3+2\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_1\Lambda_2^3\Lambda_3^3)\\
&+U^{18}(2\Lambda_1^2\Lambda_2^3\Lambda_3^2+2\Lambda_1\Lambda_2^2\Lambda_3^3+2\Lambda_1\Lambda_2^3\Lambda_3^3\\
&+\Lambda_1^2\Lambda_2^4\Lambda_3^2+\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_2^2\Lambda_3^4+\Lambda_1^2\Lambda_2^2\Lambda_3^2)\\
&+U^{19}(\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_1\Lambda_2^3\Lambda_3^3)+U^{20}(\Lambda_1\Lambda_2^4\Lambda_3^3+\Lambda_1^2\Lambda_2^2\Lambda_3^4+\Lambda_2^3\Lambda_3^4+\Lambda_1\Lambda_2^3\Lambda_3^3)\\
&+U^{21}(\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_2^2\Lambda_3^4+2\Lambda_1^2\Lambda_2^2\Lambda_3^4+2\Lambda_2^3\Lambda_3^4\\
&+2\Lambda_1\Lambda_2\Lambda_3^5+2\Lambda_1\Lambda_2^2\Lambda_3^5+\Lambda_1^2\Lambda_2^3\Lambda_3^4+\Lambda_2^4\Lambda_3^4+\Lambda_2\Lambda_3^6)\\
&+U^{22}(\Lambda_1\Lambda_2^3\Lambda_3^3+\Lambda_1\Lambda_2^2\Lambda_3^5+\Lambda_1\Lambda_2^4\Lambda_3^3+\Lambda_1\Lambda_2^3\Lambda_3^5+\Lambda_1^2\Lambda_2^2\Lambda_3^4+\Lambda_2^3\Lambda_3^4)\\
&+U^{23}(\Lambda_1\Lambda_2^2\Lambda_3^5+\Lambda_1\Lambda_2^3\Lambda_3^5+\Lambda_2^2\Lambda_3^6)+U^{24}(\Lambda_1^2\Lambda_2^3\Lambda_3^4+\Lambda_2^4\Lambda_3^4)\\
&+U^{25}(\Lambda_1\Lambda_2^2\Lambda_3^5+\Lambda_1\Lambda_2^3\Lambda_3^5+\Lambda_2^2\Lambda_3^6)+U^{26}(\Lambda_1\Lambda_2^2\Lambda_3^5+2\Lambda_1\Lambda_2^3\Lambda_3^5+\Lambda_1\Lambda_2^4\Lambda_3^5)\\
&+U^{27}(\Lambda_2^3\Lambda_3^6+\Lambda_1^2\Lambda_2^2\Lambda_3^6)+U^{28}(\Lambda_1\Lambda_2^3\Lambda_3^5+\Lambda_1\Lambda_2^4\Lambda_3^5)\\
&+U^{29}\Lambda_1^2\Lambda_2^3\Lambda_3^6+U^{31}\Lambda_1^2\Lambda_2^3\Lambda_3^6+U^{33}\Lambda_1^2\Lambda_2^4\Lambda_3^6]\\
&+[(1-U^3\Lambda_1^2)(1-U^5\Lambda_3^2)(1-U^7\Lambda_3^2)(1-U^3\Lambda_1\Lambda_3)]^{-1}
\end{aligned}
$$

$$\times[U^6(\Lambda_1\Lambda_3+2\Lambda_1\Lambda_2\Lambda_3+\Lambda_1^2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3)+U^7(\Lambda_1^2\Lambda_3^2+\Lambda_1^2\Lambda_2\Lambda_3^2)$$
$$+U^8\Lambda_1^2\Lambda_2\Lambda_3^2+U^{10}\Lambda_1^2\Lambda_2\Lambda_3^2$$
$$+U^{11}(\Lambda_1^2\Lambda_3^2+\Lambda_1^3\Lambda_3^3+\Lambda_1^3\Lambda_2\Lambda_3^3+2\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1^2\Lambda_2^2\Lambda_3^2)$$
$$+U^{12}(\Lambda_1^2\Lambda_2^2\Lambda_3^2+\Lambda_1^3\Lambda_3^3)$$
$$+U^{13}(\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1^3\Lambda_3^3+2\Lambda_1^3\Lambda_2\Lambda_3^3+\Lambda_1^2\Lambda_2^2\Lambda_3^2+\Lambda_1^3\Lambda_2^2\Lambda_3^3)$$
$$+U^{14}\Lambda_1^3\Lambda_2\Lambda_3^3+U^{16}\Lambda_1^3\Lambda_2\Lambda_3^3+U^{17}(\Lambda_1^3\Lambda_2^2\Lambda_3^3+\Lambda_1^3\Lambda_2\Lambda_3^3)$$
$$+U^{18}(2\Lambda_1^4\Lambda_2\Lambda_3^4+\Lambda_1^3\Lambda_2^2\Lambda_3^3+\Lambda_1^4\Lambda_2^2\Lambda_3^4+\Lambda_1^4\Lambda_3^4)]\}. \tag{5.3}$$

The generating function for tensors in the enveloping algebra of O(7) is found with the help of the chain SU(21) ⊃ SU(7) ⊃ O(7) for one-rowed representations of SU(21); the insertions, as mentioned in § 2, are defined by (10 . . . 0) ⊃ (010000) ⊃ (010). We found it useful to introduce the additional group SU(6) between SU(7) and O(7); the insertion is defined by (010000) ⊃ (10000) + (01000) with (10000) > (100) − (000) and (01000) > (010) − (100) + (000). O(7) is not a subgroup of SU(6), but may be subjoined to it (Patera and Sharp 1980).

The generating function for SU(21) ⊃ SU(7) branching rules (one-rowed representations) is

$$\mathscr{F}(U;K_2,K_4,K_6)=[(1-UK_2)(1-U^2K_4)(1-U^3K_6)]^{-1}. \tag{5.4}$$

U carries the SU(21) label (the degree), while K_2, K_4, K_6 carry respectively the second, fourth and sixth SU(7) labels. Thus we need the generating function for SU(7) ⊃ O(7) branching rules only for SU(7) representations with the first, third and fifth labels zero. The generating function for SU(7) ⊃ SU(6) branching rules is (odd SU(7) labels zero)

$$\mathscr{H}(K_2,K_4,K_6;M_1,M_2,M_3,M_4,M_5)$$
$$=[(1-K_2M_1)(1-K_2M_2)(1-K_4M_3)(1-K_4M_4)(1-K_6M_5)(1-K_6)]^{-1}. \tag{5.5}$$

Hence, once the generating function $\mathscr{J}(M_1,M_2,M_3,M_4,M_5;\Lambda_1,\Lambda_2,\Lambda_3)$ for SU(6) > O(7) branching rules is known (Λ_1, Λ_2, Λ_3 carry the O(7) representation labels), the SU(7) ⊃ O(7) generating function is, according to (5.5),

$$\mathscr{K}(K_2,K_4,K_6;\Lambda_1,\Lambda_2,\Lambda_3)=(1-K_6)^{-1}\mathscr{J}(K_2,K_2,K_4,K_4,K_6;\Lambda_1,\Lambda_2,\Lambda_3) \tag{5.6}$$

and, according to (5.4), the desired generating function for tensors in the enveloping algebra of O(7) is

$$\mathscr{G}(U;\Lambda_1,\Lambda_2,\Lambda_3)$$
$$=\mathscr{K}(U,U^2,U^3;\Lambda_1,\Lambda_2,\Lambda_3)$$
$$=(1-U^3)^{-1}\mathscr{J}(U,U,U^2,U^2,U^3;\Lambda_1,\Lambda_2,\Lambda_3). \tag{5.7}$$

For brevity the functions $\mathscr{J}$ and $\mathscr{K}$ are omitted here; they can be found in Couture (1980).

The result for $\mathscr{G}(U;\Lambda_1,\Lambda_2,\Lambda_3)$ is

$$\mathscr{G}(U;\Lambda_1,\Lambda_2,\Lambda_3)$$
$$=[(1-U^2)(1-U^4)(1-U^6)(1-U\Lambda_2)(1-U^2\Lambda_1^2)(1-U^2\Lambda_3^2)(1-U^3\Lambda_1)$$

$$
\begin{aligned}
&\times(1-U^4\Lambda_2^2)]^{-1}\{[(1-U^3\Lambda_2)(1-U^4\Lambda_1^2)(1-U^5\Lambda_2)(1-U^6\Lambda_3^2)]^{-1}\\
&\times[1+U^6\Lambda_1^2\Lambda_2+U^7\Lambda_1\Lambda_2+U^8\Lambda_2\Lambda_3^2+U^9(\Lambda_1\Lambda_2^2+\Lambda_1\Lambda_2\Lambda_3^2)\\
&+U^{10}\Lambda_1\Lambda_2\Lambda_3^2+U^{11}\Lambda_1^2\Lambda_2\Lambda_3^2]+[(1-U^4\Lambda_1^2)(1-U^5\Lambda_2)(1-U^4\Lambda_3^2)\\
&\times(1-U^6\Lambda_3^2)]^{-1}[U^4\Lambda_3^2+U^6(\Lambda_1\Lambda_3^2+\Lambda_2\Lambda_3^2)+U^8\Lambda_1\Lambda_2\Lambda_3^2\\
&+U^9\Lambda_1^2\Lambda_3^2+U^{12}\Lambda_1\Lambda_2\Lambda_3^2+U^{13}\Lambda_1\Lambda_2\Lambda_3^4+U^{15}\Lambda_1^2\Lambda_2\Lambda_3^4]\\
&+[(1-U^4\Lambda_3^2)(1-U^5\Lambda_2)(1-U^6\Lambda_3^2)(1-U^8\Lambda_2^2)]^{-1}\\
&\times[U^9\Lambda_1\Lambda_2\Lambda_3^2+U^{10}\Lambda_2\Lambda_3^2+U^{11}\Lambda_1\Lambda_2^2\Lambda_3^2+U^{12}\Lambda_2^2\Lambda_3^2\\
&+U^{14}\Lambda_2^3\Lambda_3^2+U^{15}\Lambda_1\Lambda_2^2\Lambda_3^2+U^{16}\Lambda_2^2\Lambda_3^4+U^{21}\Lambda_1\Lambda_2^3\Lambda_3^4]\\
&+[(1-U^4\Lambda_1^2)(1-U^4\Lambda_3^2)(1-U^6\Lambda_3^2)(1-U^6\Lambda_3^4)]^{-1}\\
&\times[U^8\Lambda_1\Lambda_3^4+U^{10}(\Lambda_1\Lambda_2\Lambda_3^4+\Lambda_3^6)+2U^{11}\Lambda_1\Lambda_3^4+U^{12}(\Lambda_1\Lambda_3^6+\Lambda_2\Lambda_3^6)\\
&+U^{13}(\Lambda_1\Lambda_2\Lambda_3^4+\Lambda_3^6)+U^{14}\Lambda_1\Lambda_2\Lambda_3^6+U^{15}(\Lambda_2\Lambda_3^6+\Lambda_1^2\Lambda_3^6)+U^{16}\Lambda_1^2\Lambda_3^6\\
&+U^{17}\Lambda_1^2\Lambda_2\Lambda_3^6+U^{18}\Lambda_1^2\Lambda_2\Lambda_3^6+U^{19}\Lambda_1\Lambda_2\Lambda_3^8]\\
&+[(1-U^3\Lambda_2)(1-U^6\Lambda_3^2)(1-U^6\Lambda_3^4)(1-U^8\Lambda_2^2)]^{-1}\\
&\times[U^7\Lambda_2\Lambda_3^2+U^9\Lambda_2^2\Lambda_3^2+U^{11}\Lambda_1\Lambda_2\Lambda_3^4+U^{12}\Lambda_1\Lambda_2^2\Lambda_3^2\\
&+U^{13}(\Lambda_2^2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3^4)+U^{14}(\Lambda_2^2\Lambda_3^4+\Lambda_1\Lambda_2^3\Lambda_3^2)+U^{16}\Lambda_2^3\Lambda_3^4\\
&+U^{17}\Lambda_1\Lambda_2^2\Lambda_3^4+U^{18}\Lambda_1\Lambda_2^3\Lambda_3^2+U^{19}\Lambda_1\Lambda_2^3\Lambda_3^4+U^{20}\Lambda_2^3\Lambda_3^4+U^{21}\Lambda_2^3\Lambda_3^6\\
&+U^{22}\Lambda_2^4\Lambda_3^4+U^{26}\Lambda_1\Lambda_2^4\Lambda_3^6]+[(1-U^4\Lambda_3^2)(1-U^6\Lambda_3^2)(1-U^6\Lambda_3^4)\\
&\times(1-U^8\Lambda_2^2)]^{-1}[U^{11}\Lambda_2\Lambda_3^4+U^{13}\Lambda_2^2\Lambda_3^4+U^{14}\Lambda_1\Lambda_2\Lambda_3^4\\
&+U^{15}\Lambda_1\Lambda_2\Lambda_3^6+U^{16}(\Lambda_2\Lambda_3^6+2\Lambda_1\Lambda_2^2\Lambda_3^4)+U^{17}(\Lambda_2^2\Lambda_3^4+\Lambda_1\Lambda_2^2\Lambda_3^6)\\
&+U^{18}(\Lambda_1\Lambda_2^3\Lambda_3^4+2\Lambda_2^2\Lambda_3^6)+U^{19}\Lambda_2^3\Lambda_3^4+U^{20}\Lambda_2^3\Lambda_3^6+U^{21}\Lambda_1\Lambda_2^2\Lambda_3^6\\
&+U^{23}\Lambda_1\Lambda_2^3\Lambda_3^6]+[(1-U^3\Lambda_2)(1-U^5\Lambda_2)(1-U^6\Lambda_3^2)(1-U^8\Lambda_2^2)]^{-1}\\
&\times[U^5\Lambda_1\Lambda_2+U^8\Lambda_2^2+U^{12}(\Lambda_2^3+\Lambda_2^2\Lambda_3^2)+U^{13}\Lambda_1\Lambda_2^2\Lambda_3^2+U^{16}\Lambda_2^3\Lambda_3^2\\
&+U^{17}(\Lambda_1\Lambda_2^4+\Lambda_1\Lambda_2^3\Lambda_3^2)]+[(1-U^3\Lambda_1\Lambda_3^2)(1-U^4\Lambda_1^2)(1-U^4\Lambda_3^2)\\
&\times(1-U^6\Lambda_3^4)]^{-1}[U^3\Lambda_1\Lambda_3^2+U^5(\Lambda_1\Lambda_3^2+\Lambda_1\Lambda_2\Lambda_3^2)+U^7(\Lambda_1^2\Lambda_3^4+\Lambda_1\Lambda_2\Lambda_3^2)\\
&+U^9(2\Lambda_1^2\Lambda_3^4+\Lambda_1^2\Lambda_2\Lambda_3^4)+U^{10}\Lambda_1^2\Lambda_3^4+U^{11}(\Lambda_1^2\Lambda_3^4+2\Lambda_1^2\Lambda_2\Lambda_3^4)\\
&+U^{12}\Lambda_1^2\Lambda_2\Lambda_3^4+U^{13}\Lambda_1^2\Lambda_2\Lambda_3^4+U^{15}\Lambda_1^3\Lambda_3^6+U^{17}\Lambda_1^3\Lambda_2\Lambda_3^6]\\
&+[(1-U^3\Lambda_2)(1-U^4\Lambda_1^2)(1-U^6\Lambda_3^2)(1-U^6\Lambda_3^4)]^{-1}\\
&\times[U^4\Lambda_1\Lambda_3^2+U^6(\Lambda_1\Lambda_2\Lambda_3^2+\Lambda_3^4)+U^7\Lambda_1\Lambda_3^2\\
&+U^8(\Lambda_1\Lambda_3^2+\Lambda_2\Lambda_3^4+\Lambda_1\Lambda_2\Lambda_3^2)+U^9\Lambda_3^4+U^{10}\Lambda_1\Lambda_2^2\Lambda_3^2+U^{11}\Lambda_2\Lambda_3^4\\
&+U^{12}(\Lambda_1^2\Lambda_3^4+\Lambda_1^2\Lambda_2\Lambda_3^4)+U^{14}(\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_1^2\Lambda_2^2\Lambda_3^4)\\
&+U^{15}\Lambda_1\Lambda_2\Lambda_3^6+U^{16}\Lambda_1\Lambda_2\Lambda_3^6]\}.
\end{aligned}
\tag{5.8}
$$

The integrity basis corresponding to the generating function (5.3) is found in Couture (1980).

6. Testing the results

In § 2 we discussed two methods of obtaining the generating function for tensors in the enveloping algebra of a group. The first (working through a subgroup) is an analytic derivation and constitutes a rigorous mathematical proof; this approach was used for SU(3) and O(5). The second approach, which makes use of a larger group, involves finding a set of elementary multiplets (the integrity basis) and relations among them (syzygies). It does not constitute a proof, since one is not sure that all elementary multiplets and relations have been found; in this case the result must be checked.

Our generating functions satisfy the obvious requirements. There are $\frac{1}{2}(r+l)$ denominator factors, including l which correspond to Casimir invariants; the highest degree with which any tensor appears is that prescribed by Kostant's (1963) theorem (1.1). Each generating function is consistent with known generating functions for subgroup scalars in the enveloping algebra. Indeed, the above checks were often helpful in determining the generating functions.

The most conclusive check which we apply is the reduction of the generating function for tensors in the enveloping algebra to the corresponding generating function for weights. In what follows we use Sp(6) as an example.

The generating function for tensors may be reduced to that for weights by substituting the character generator of the group; unfortunately, the character generator is not known in general, so one must work through a chain of subgroups. For Sp(6) the following chain is convenient:

$$\mathrm{Sp}(6)\supset \mathrm{Sp}(4)\times \mathrm{SU}(2)\supset \mathrm{SU}(2)\times \mathrm{U}(1)\times \mathrm{SU}(2)\supset \mathrm{U}(1)\times \mathrm{U}(1)\times \mathrm{U}(1).$$

The generating function for branching rules at each stage is known. For $\mathrm{Sp}(6)\supset \mathrm{Sp}(4)\times \mathrm{SU}(2)$ we have (Sharp 1970)

$$\begin{aligned}&[(1-\Lambda_1N_1)(1-\Lambda_1N_3)(1-\Lambda_2N_2)(1-\Lambda_2)(1-\Lambda_3N_1)(1-\Lambda_3N_2N_3)]^{-1}\\ &\qquad\times[(1-\Lambda_2N_1N_3)^{-1}+\Lambda_1\Lambda_3N_2(1-\Lambda_1\Lambda_3N_2)^{-1}],\end{aligned}\tag{6.1}$$

where Λ_1, Λ_2, Λ_3 carry the Sp(6) labels, N_1, N_2 the Sp(4) labels, and N_3 carries the SU(2) label (the dimension of the SU(2) representation (ν) is $\nu+1$). The generating function for $\mathrm{Sp}(4)\supset \mathrm{SU}(2)\times \mathrm{U}(1)$ is (Sharp and Lam 1969)

$$\begin{aligned}&[(1-N_1N_4\eta_3)(1-N_1N_4\eta_3^{-1})(1-N_2\eta_3^2)(1-N_2\eta_3^{-2})]^{-1}\\ &\qquad\times[(1-N_1^2)^{-1}+N_2N_4^2(1-N_2N_4^2)^{-1}],\end{aligned}\tag{6.2}$$

where N_4 carries the SU(2) label and η_3 the U(1) label. The generating function for $\mathrm{SU}(2)\supset \mathrm{U}(1)$ is

$$[(1-N\eta)(1-N\eta^{-1})]^{-1},\tag{6.3}$$

where N carries the SU(2) label and η the U(1) label (weight). Substitution of (6.1) into (5.3), of (6.2) into the result, and finally of (6.3) into that result, converts (5.3) into a generating function for weights in the Sp(6) enveloping algebra. But the actual generating function for weights is

$$\begin{aligned}&[(1-U\eta_2^2\eta_3^2)(1-U\eta_3^2)(1-U\eta_2^{-2}\eta_3^2)(1-U\eta_2^2\eta_3^{-2})(1-U\eta_2^{-2}\eta_3^{-2})\\ &\qquad\times(1-U\eta_3^{-2})(1-U\eta_1^2)(1-U\eta_1^{-2})(1-U\eta_2^2)(1-U\eta_2^{-2})(1-U\eta_1\eta_2\eta_3)\end{aligned}$$

$$\times(1-U\eta_1\eta_2^{-1}\eta_3)(1-U\eta_1^{-1}\eta_2\eta_3)(1-U\eta_1^{-1}\eta_2^{-1}\eta_3)(1-U\eta_1\eta_2\eta_3^{-1})$$
$$\times(1-U\eta_1\eta_2^{-1}\eta_3^{-1})(1-U\eta_1^{-1}\eta_2\eta_3^{-1})(1-U\eta_1^{-1}\eta_2^{-1}\eta_3^{-1})(1-U)^3]^{-1}. \quad (6.4)$$

Each factor in (6.4) corresponds to a weight in the adjoint representation of Sp(6).

The substitutions described above can, in principle, be done analytically, but the algebra soon gets out of hand. We have written a computer program which performs the substitutions numerically and compares the result with (6.4). The comparison can be made with high precision, for random values of the dummy variables on which the generating function depends. With quadruple precision the relative error was of order 10^{-23} for Sp(6). To check the efficacy of the numerical comparison, we made minimal changes in the generating function being tested, such as altering by unity a coefficient or exponent; such a change increases the relative error by many orders of magnitude.

The reduction of the generating function to that for weights must utilise a chain of groups of equal rank at each stage; otherwise information is lost. We applied similar numerical checks to all the generating functions in this paper.

7. Discussion

In this section we expand briefly on a few points mentioned in the introduction.

First we illustrate how to find the algebraic form of the elementary tensors in the enveloping algebra, with SU(3) as an example. According to (3.6) the elementary tensors are (1, 11), (2, 00), (2, 11), (3, 00), (3, 30) and (3, 03); the first number is the degree, the other two the representation labels of the tensor. We content ourselves with finding the highest weight component of each. For brevity we denote the components of the basic octet by letters, $\alpha=|1,\frac{1}{2},\frac{1}{2}\rangle$, $\beta=|1,\frac{1}{2},-\frac{1}{2}\rangle$, $\gamma=|0,1,1\rangle$, $\delta=|0,1,0\rangle$, $\epsilon=|0,1,-1\rangle$, $\theta=|0,0,0\rangle$, $\kappa=|-1,\frac{1}{2},\frac{1}{2}\rangle$, $\lambda=|-1,\frac{1}{2},-\frac{1}{2}\rangle$; the notation is $|Y,T,T_0\rangle$. For each elementary tensor, write the highest component as an unknown linear combination of those monomials in $\alpha,\ldots,\lambda$ which have the necessary degree and weight. The coefficients are found by requiring that the generators corresponding to the simple roots annihilate the component. For SU(3) these simple generators are E_{12} and E_{23} in the notation of Gel'fand and Zetlin (1950); we adopt Gel'fand and Zetlin's matrix elements of E_{12} and E_{23} between the components of the octet. Then we find

$$\begin{aligned}
(1,11)&\sim\alpha,\\
(2,00)&\sim\alpha\lambda-\beta\kappa-\gamma\epsilon+\tfrac{1}{2}\delta^2+\tfrac{1}{2}\theta^2,\\
(2,11)&\sim\sqrt{3}\alpha\delta-\sqrt{6}\beta\gamma+\alpha\theta,\\
(3,00)&\sim\sqrt{3}\alpha\delta\lambda+\sqrt{3}\beta\delta\kappa-\sqrt{6}\beta\gamma\lambda-\sqrt{6}\alpha\epsilon\kappa+\alpha\theta\lambda-\beta\theta\kappa-\delta^2\theta+2\gamma\epsilon\theta+\tfrac{1}{3}\theta^3,\\
(3,30)&\sim\alpha\gamma\delta+\sqrt{3}\alpha\gamma\theta-\sqrt{2}\alpha^2\kappa-\sqrt{2}\beta\gamma^2,\\
(3,03)&\sim\alpha^2\epsilon+\beta^2\gamma-\sqrt{2}\alpha\beta\gamma.
\end{aligned} \quad (7.1)$$

The highest component of any tensor in the enveloping algebra is a product of powers of the elementary factors (7.1) ((3, 30) and (3, 03) should not appear in the same product).

We turn to the question of subgroup scalars in the enveloping algebra of a group. Besides the Casimir operators of group and subgroup, there are $r_G-l_G-r_H-l_H$ functionally independent subgroup scalars, or missing label operators (twice the number actually needed to resolve the labelling problem (Peccia and Sharp 1976); r_G, r_H, l_G, l_H are the order and rank of group and subgroup. The generating function $\mathscr{G}(U; \Lambda_1, \Lambda_2, \ldots)$ for tensors in the enveloping algebra contains information about subgroup scalars. Substitute into $\mathscr{G}$ the generating function $\mathscr{F}(\Lambda_1, \Lambda_2, \ldots)$ for subgroup scalars in representations of the group; there results the generating function $\mathscr{H}(U)$ for subgroup scalars in the enveloping algebra. This substitution is often very simple to make.

As a first example consider SU(3) ⊃ O(3); the generating function for O(3) scalars in SU(3) representations is

$$\mathscr{F}(\Lambda_1, \Lambda_2) = [(1-\Lambda_1^2)(1-\Lambda_2^2)]^{-1}, \tag{7.2}$$

obtained by setting equal to zero the dummy which carries the O(3) representation label in the SU(3) ⊃ O(3) branching rules generating function, given, for example, in I. (7.2) states that each even–even representation of SU(3) contains one O(3) scalar. Substitution of (7.2) in (3.6) means keeping the part of (3.6) even in Λ_1 and in Λ_2 and then setting $\Lambda_1 = \Lambda_2 = 1$. The result is

$$\mathscr{H}(U) = (1+U^6)[(1-U^2)^2(1-U^3)^2(1-U^4)]^{-1}, \tag{7.3}$$

which agrees with the generating function of Judd *et al* (1974) when their dummy variables D and P are set equal to U.

Similarly, the generating function for SU(3) scalars in the G_2 enveloping algebra is obtained from (4.9) by setting $\Lambda_2 = 0$, $\Lambda_1 = 1$; that for SU(2) × SU(2) scalars in the SU(4) enveloping algebra is obtained from (4.2) by setting $\Lambda_2 = 0$, keeping the part even in Λ_1 and in Λ_3 and setting $\Lambda_1 = \Lambda_3 = 1$; that for SU(2) × U(1) scalars in the O(5) enveloping algebra is found from (3.8) by keeping the part even in Λ_1 and in Λ_2 and setting $\Lambda_1 = \Lambda_2 = 1$; that for G_2 scalars in the O(7) enveloping algebra is obtained from (5.8) by setting $\Lambda_1 = \Lambda_2 = 0$, $\Lambda_3 = 1$. The resulting generating functions can be compared with the generating functions or integrity bases given, variously, by Quesne (1976), Sharp (1975), and in I. Many new generating functions for subgroup scalars could be found in this way. Of particular interest, because of its importance in nuclear physics, is the group–subgroup Sp(6) ⊃ SU(3) × U(1) (Rosensteel and Rowe 1977); the generating function for subgroup scalars is obtained from (5.3) by retaining the part even in Λ_1, in Λ_2 and in Λ_3 and setting $\Lambda_1 = \Lambda_2 = \Lambda_3 = 1$.

Finally we touch on the collapse of the generating function for tensors in the enveloping algebra when the generators act only on restricted representations for which one or more Cartan labels vanish. We hope to do more work on higher groups, but here we report on SU(3) as a simple example.

The generators, acting on states transforming by the special representations $(\lambda_1, 0)$, are represented by simple differential operators. In the notation used earlier in this section we have

$$\begin{aligned} &\alpha \to \eta\, \partial_\zeta, \quad \beta \to \xi\, \partial_\zeta, \quad \gamma \to -\eta\, \partial_\xi, \quad \delta \to 2^{-1/2}(\eta\, \partial_\eta - \xi\, \partial_\xi), \\ &\epsilon \to \xi\, \partial_\eta, \quad \theta \to 6^{-1/2}(-\eta\, \partial_\eta - \xi\, \partial_\xi + 2\zeta\, \partial_\zeta), \quad \kappa \to -\zeta\, \partial_\xi, \quad \lambda \to \zeta\, \partial_\eta. \end{aligned} \tag{7.4}$$

This approach was suggested by M Moshinsky (private communication). With these substitutions made in (7.1), and the operators all symmetrised as to ordering of generators, several relations appear between the generators. The elementary tensors (3, 30) and (3, 03) vanish. A linearly independent set of tensors in the enveloping algebra consists of the stretched tensor products $(1, 00)^a \times (1, 11)^b$, where a and b are any non-negative integers, except that $a=1$, $b=0$ is excluded. Thus the collapsed generating function is

$$\frac{1}{(1-U)(1-U\Lambda_1\Lambda_2)}-U=\frac{1+U^2\Lambda_1\Lambda_2}{(1-U^2)(1-U\Lambda_1\Lambda_2)}+\frac{U^3}{1-U^2}. \tag{7.5}$$

The scalar (1, 00), denoted in (7.5) by U, has as its eigenvalue the representation label λ_1. Modulo multiplication by a scalar, the adjoint representation appears just once in (7.5), of degree 1, in agreement with a result of Okubo (1977).

Acknowledgments

We are grateful for helpful discussions with R Gilmore, M Moshinsky, J Patera and R P Stanley, and to a referee for useful criticisms. The branching tables of McKay and Patera (1980) were of great assistance.

Appendix

We define the group–subgroup characteristic function for the group–subgroup G $\supset$ H as

$$\xi_\lambda^{\mathrm{H}}(\eta)=(\Delta(\eta)/\Delta'(\eta))\sum_\nu c_{\lambda\nu}\eta^{K_\nu}. \tag{A1}$$

The symbols in (A1) must now be defined.

$c_{\lambda\nu}$ is the multiplicity of the H representation ν in the G representation λ.

η^{K_ν} means $\Pi_i\,\eta_i^{(K_\nu)_i}$, where $(K_\nu)_i$ is the ith component of the vector

$$\boldsymbol{K}_\nu=\boldsymbol{R}+\boldsymbol{W}_\nu; \tag{A2}$$

$\boldsymbol{R}$ is half the sum of the positive roots of H, and $\boldsymbol{W}_\nu$ is the highest weight of the representation ν.

$$\Delta'(\eta)=\sum_S(-1)^S\eta^{SR} \tag{A3}$$

is Weyl's characteristic function for the scalar representation of H; the sum is over Weyl reflections S, and $(-1)^S$ is the determinant of the matrix of S.

Similarly $\Delta(\eta)$ is Weyl's characteristic function for the scalar representation of G in which (if necessary) a projection onto the weight space of H has been effected by substituting for the variables in terms of the η appropriate to H.

The constructive definition (A1) permits the evaluation of $\xi_\lambda^{\mathrm{H}}(\eta)$ for any representation λ for which the branching multiplicities $c_{\lambda\nu}$ are known. In particular, for the scalar representation, $c_{0\nu}=\delta_{0\nu}$ and

$$\xi_0^{\mathrm{H}}(\eta)=\Delta(\eta)\eta^R/\Delta'(\eta). \tag{A4}$$

Dividing (A1) by (A4) we obtain

$$\xi_\lambda^{\mathrm{H}}(\eta)/\xi_0^{\mathrm{H}}(\eta)=\sum_\nu c_{\lambda\nu}\eta^{W_\nu}. \tag{A5}$$

If one substitutes for the variables η in terms of new variables N so that $\eta^{W_\nu}=N^\nu$, the result is equation (2.2).

We now discuss some properties of the group–subgroup characteristic function. First we sketch a proof that $\xi_\lambda^{\mathrm{H}}(\eta)$ is a sum of monomials. Weyl (1926) shows that the vectors ***SR*** for any compact Lie group are possible weights of that group. After projection onto the weight space of a subgroup, they will be possible subgroup weights. From this it can be shown that $\Delta(\eta)$ is a linear combination of Weyl characteristic functions $\xi_\nu(\eta)$ of H, each of which is, of course, divisible by $\Delta'(\eta)$. Since $\Delta(\eta)/\Delta'(\eta)$ is a sum of monomials, it follows from (A1) that $\xi_\lambda^{\mathrm{H}}(\eta)$ is also a sum of monomials.

We can say something about the distribution of the terms of $\xi_\lambda^{\mathrm{H}}(\eta)$ in weight space. From the form of (A1) it is clear that they lie in or near the dominant sector of H weight space, the sector of highest weights of H representations (the terms of Δ/Δ' are independent of λ and cannot shift them far). Weyl's (1926) characteristic function for the subgroup H is

$$\xi_\nu(\eta)=\sum_S(-1)^S\eta^{SK_\nu}. \tag{A6}$$

In terms of it the character function is

$$\chi_\nu(\eta)=\xi_\nu(\eta)/\Delta'(\eta). \tag{A7}$$

The symbols are defined as in (A2) and (A3). There are similar equations for the group G. From the additivity of the characters under the reduction G to H,

$$\chi_\lambda(\eta)=\sum_\nu c_{\lambda\nu}\chi_\nu(\eta), \tag{A8}$$

and (A6), (A7) we obtain

$$\xi_\lambda(\eta)=(\Delta/\Delta')\sum_\nu c_{\lambda\nu}\xi_\nu(\eta), \tag{A9}$$

which incidentally suggests an efficient way of calculating branching rules (just divide ξ_λ by Δ/Δ' and retain the part of the quotient in the dominant sector of subgroup weight space). Now the number of terms in $\xi_\lambda(\eta)$ is fixed (independent of λ), and they all lie equidistant from the origin of weight space, at least before projection onto H weight space; much cancellation occurs between the terms on the right-hand side of (A9). Now our group–subgroup characteristic function (A1) differs from (A9) only in that it lacks the sum over Weyl reflections S implicit in the definition (A6) of $\xi_\nu(\eta)$. Hence most of the cancellation in (A9) persists, since the parts of (A9) coming from different sectors (under Weyl reflections of H) cannot cancel mutually except near the boundaries of the sectors, because of the small shifts due to Δ/Δ'. We can conclude that the terms of $\xi_\lambda^{\mathrm{H}}(\eta)$ are either terms from $\xi_\lambda(\eta)$ which project into the dominant H sector, or else lie on or near the boundaries of the dominant H sector.

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